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Exact Solutions to the Time-Dependent Lorentz Gas Boltzmann Equation: The Approach to Hydrodynamics

John Palmeri[1,2]

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New exact solutions to the time-dependent Lorentz gas Boltzmann equation are presented for two classes of nonequilibrium initial value problems: the *decay of localized disturbances* and the *response to applied electric fields*. These exact results are used to gain some insight into the crossover of the nonequilibrium state from the early-time *kinetic* regime to the late-time *hydrodynamic* regime.

KEY WORDS: Transport theory; Lorentz gas; time-dependent Boltzmann equation; exact results.

1. INTRODUCTION

In many-body systems transport or kinetic equations describing the behavior of the one-particle distribution function are often the centerpiece in theoretical studies of nonequilibrium phenomena in systems as diverse as ordinary gases, neutron chain reactors, quantum liquids, and QCD plasmas, to name just a few.[1,2] Transport equations are also often the starting point in attempts to attain one of the principal goals of statistical mechanics, namely an explanation of the macroscopic properties of matter from a microscopic description. In most cases useful analytical solutions to these equations remain out of reach, primarily due to the presence of a collision integral that accounts for the scattering of the particles. On the other hand, the few cases where exact results can be obtained serve as a touchstone for one's ideas about transport phenomena in general.

[1] Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801.
[2] Current address: CEA-CEN de Saclay, Service de Physique Théorique, F-91191 Gif-sur-Yvette Cedex, France.

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Even solving these equations numerically is difficult,[3] especially for the class of time-dependent initial-value transport problems considered here: *the approach to hydrodynamics following a pulsed, localized disturbance* and in the case of a charged system *the response to a turned on, applied electric field.* For example, during the decay of a localized perturbation there will be wavefronts propagating through the system, and numerical schemes are hard-pressed to give accurate results in regions near the wavefront where the distribution function is varying rapidly. The wavefronts should only play an important role for times $t \lesssim \tau$ [the mean free time (MFT) for scattering] following the *detonation* of such a localized disturbance, for at such times the system is in the so-called *kinetic regime* where a diffusive or hydrodynamic description, which coarse grains over processes at fine time and length scales, breaks down completely. In this regime a solution to the full kinetic equation is essential. On the other hand, for times $t \gg \tau$ the transport transients have died out, and the system can be well described by the more tractable macroscopic hydrodynamic equations. There are important physical problems in, for example, such fields as quantum liquids,[4] semiconductors,[5] high-energy heavy-ion collisions,[6] and stellar astrophysics,[7] where getting a handle on the behavior of the system in the difficult kinetic regime following a pulsed, localized perturbation is of great interest.

Another important class of nonequilibrium problems (especially in semiconductor physics) is concerned with the response of a gas of charged carriers to an applied external electric field. The purpose of this paper is to study these two classes of important nonequilibrium problems in a model system simple enough so that exact results can be obtained, but sophisticated enough so that perhaps some light can be shed on similiar phenomena in more realistic systems.

The model studied here is the 3D Lorentz gas, a system of noninteracting particles scattering (elastically) off of an infinite random array of fixed spherical scatterers.[1,8,9] This model is closely allied with random walk problems and is a classic in theoretical physics. It has been useful in elucidating the nature of contracted (hydrodynamic) descriptions of transport processes (the Chapman–Enskog expansion) and in helping to resolve the famous Hilbert paradox. Because the Boltzmann equation (BE) for the Lorentz gas contains a true *collision integral,* one hopes that many features of transport in this relatively simple model system are in some sense generic (see refs. 8 and 9 and Section 4 below). One justification for exploring the details of such a simplified model is that transport properties seem to depend in many ways on the *existence* of collisions, not their exact nature.

It is important to keep in mind that the BE for a dilute *interacting gas* contains two-body scattering and is therefore nonlinear. Although the results for the (linear) Lorentz gas BE cannot be used to say much about the nonlinear features of the dilute gas problem, they may be able to be applied to the linearized version of the dilute gas BE. Unfortunately, the two linear BEs are not exactly the same, and the differences make a direct carryover of the Lorentz gas results somewhat problematic (see, e.g., pp. 174 and 311 of ref. 1).

Strictly speaking, the Boltzmann equation itself is only valid in the so-called Boltzmann–Grad limit, where the density of scatterers $\rho \to \infty$ while the radius $R \to 0$ in such a way that the mean free path $\sim (\rho R^2)^{-1} \to \text{const.}$ In this limit the scattering becomes a Markovian process, which allows one to "derive" the Boltzmann equation. In general, the scattering is non-Markovian, and the memory effects that follow are studied in molecular dynamical simulations.

A special feature of the problems considered here is that, due to the absence of inelastic scattering, there is no steady-state solution or true approach to equilibrium (besides the spreading out of an initially localized disturbance), which renders inapplicable familiar approximations such as the relaxation-time approximation (RTA).

The localized disturbance problem is treated first. In Section 2, with the assumption of plane symmetry, exact closed-form expressions (i.e., in terms of elementary functions or relatively simple integrals of elementary functions) for the initial-value Green's functions (GFs) are derived for the Lorentz gas Boltzmann equation. Then in Section 3, by making use of these solutions, I construct density GFs for the more physically interesting initial-value problems with spherical symmetry; these GFs allow one to find the density of particles due to an arbitrary, spherically symmetric source turned on at time $t = t_0$. The spherically symmetric solutions are then evaluated for some representative initial conditions, with special attention paid to the crossover from quasiballistic transport at early times to the diffusive or hydrodynamic behavior at long times. (Non-spherically-symmetric 3D problems are more difficult to solve and are not dealt with here). At very early times $t \ll \tau$ (collisionless, or Knudsen, regime) the particles are more or less freely streaming (since very few collisions have taken place) and the disturbance moves out ballistically with the size of the disturbance $\propto t$, while, as mentioned above, at long times a hydrodynamic description is appropriate with the disturbance moving out diffusively $\propto \sqrt{t}$. It is at intermediate times $t \lesssim \tau$ that the behavior of the system is the most difficult to pin down, and a study of this *kinetic regime* forms the central focus of this paper. My goal is to see in detail how in this model problem hydrodynamics comes out of the kinetic equation.

These spherically symmetric intial-value problems in the Lorentz gas are illuminating examples of a more general transport problem common in the real world that can be formulated as follows: Describe the response of a large system (initially in uniform global equilibrium) to a localized disturbance (confined, say, to some region around the origin) triggered at time $t = t_0$. In Section 4 the exact results derived for the Lorentz gas serve as the basis for a discussion of this general nonequilibrium statistical mechanics problem.

In Section 5 it is shown that with some restrictions the problem of a homogeneous charged gas in the presence of an applied electric field can be mapped onto the localized disturbance problem. The exact solutions for the localized disturbance problem can then be used to follow the *heating* of the gas and the growth of a current following the turning on of the field (the gas is assumed to be in global equilibrium before the field is turned on).

Exact formal solutions to the fundamental time-dependent transport problems considered here have been known for a long time,[8–14] but due to their mathematical complexity and opacity it seemed impossible, or at least extremely difficult, to evaluate them numerically and therebye flesh out the qualitatively known features of the solutions. Accurate numerical results were not obtained until an exact solution in the form of an infinite sum (von Neumann or multiple collision expansion) was derived.[15,16] The derivation of the new solutions given here is not only simpler and more straightforward than previous calculations, but the results are more intuitively appealing and mathematically transparent and also more easily evaluated numerically. Furthermore, in contrast to the multiple-collision solutions, the present exact results allow a clean separation between the transport transient and the asymptotic (hydrodynamic) parts of the solution.

The original motivation for this work was a conjecture by Leggett[4] that certain nonequilibrium structures conducive to the nucleation of the superfluid ^{3}He B-phase might develop following the intense heating of a localized region in a Fermi liquid. In an effort to test this conjecture, I attempted[17] to generalize Brooker and Syke's[18] work on the decay of a homogeneous disturbance in a Fermi liquid to the general inhomogeneous case. The extreme difficulty I encountered led me to consider the simplified but still rich Lorentz gas model that I study here.

The decay of disturbances in a dilute gas is described by a nonlinear Boltzmann equation, and finding exact solutions even in the homogeneous case is extremely difficult and is only possible for some artificial model problems (see ref. 19 for a review). The inhomogeneous problem is much more difficult[19a] ("The main challenge remains the general solution of the initial value problem for the spatially non-uniform nonlinear Boltzmann equations, or perhaps only for simplified model of it, enabling us to discuss hydrodynamic phenomena"[19]) and I believe the work presented here on the approach to hydrodynamics in a nonuniform Lorentz gas is a first step in the right direction. Exact solutions of Boltzmann equations containing true collision integrals are extremely rare, and as far as I know the exact results obtained for the decay of localized disturbances in a Lorentz gas are the only exact and numerically tractable results known for this type of transport problem. These results should therefore be of mathematical as well as physical interest in that many of the qualitative features of these solutions should be present in the more difficult problems discussed above.

In its approach the present work is very much in the spirit of Landau's classic paper on solving the Vlasov equation,[20] and it is remarkable that the integral transform techniques employed here allow such a complete and numerically tractable solution to the Lorentz gas Boltzmann equation, something which has almost inexplicably gone unnoticed for a long time despite the large amount of work devoted to this problem. More background information, a more in-depth discussion of the methods used here, and a small part of the results have been described in a previous paper,[21] to which I will refer to fill in some of the technical details. In this previous paper exact expressions for the GFs were obtained, but most of these expressions were complicated by the presence of compound integrations, something which is entirely avoided here.

2. PLANE SYMMETRY SOLUTIONS TO THE LORENTZ GAS BOLTZMANN EQUATION

The (linear) Boltzmann equation governing the behavior of the one-particle distribution function $\psi(\mathbf{r}, \hat{\mathbf{p}}, t)$ (the phase space density of particles) for a Lorentz gas can be written as

$$\frac{\partial\psi}{\partial t} + v\hat{\mathbf{p}}\cdot\nabla\psi + v\sigma\psi = v\sigma \int \frac{d\hat{\mathbf{p}}'}{4\pi}\, \psi(\mathbf{r}, \hat{\mathbf{p}}', t) + S(\mathbf{r}, \hat{\mathbf{p}}, t) \tag{1}$$

where $\hat{\mathbf{p}}$ is the unit vector in the direction of the velocity, σ is the scattering cross section, v is the (constant) speed of the particles, and S is a source term. In Eq. (1) the $v\sigma\psi$ term accounts for the scattering out of the state $\hat{\mathbf{p}}$ at the space-time point $(\mathbf{r}, t)$, while the integral term provides for the population of the state $\hat{\mathbf{p}}$ due to the scattering of particles from state $\hat{\mathbf{p}}' \to \hat{\mathbf{p}}$. The only conserved quantity is the total number of particles, implying that the density $n(\mathbf{r}, t) = \int d\hat{\mathbf{p}}\, \psi(\mathbf{r}, \hat{\mathbf{p}}, t)$ is the only hydrodynamic moment for this system. The MFT that sets the time scale for collision processes is defined by $\tau \equiv 1/v\sigma$.

Since only elastic scattering processes are present in Eq. (1), the magnitude of the momentum $(p = mv)$ enters only as a parameter, and the dependence of ψ on p is suppressed (the states with different energies $E = p^2/2m$ are completely decoupled). Any function depending only on p is trivially a solution to the homogeneous version of Eq. (1) ($S \equiv 0$); thus, one can imagine that the gas is in some thermal equilibrium state $\psi_0(p)$ for $t < 0$ after which it is disturbed, so that the full distribution function can be written as $\Psi(\mathbf{r}, p, \hat{\mathbf{p}}, t) = \psi_0(p) + \psi(\mathbf{r}, p, \hat{\mathbf{p}}, t)$.

The simple RTA to the Boltzmann equation can be obtained by neglecting the integral term in Eq. (1); the BE then reduces to

$$\frac{\partial \psi}{\partial t} + v\hat{\mathbf{p}} \cdot \nabla \psi + v\sigma\psi = +S(\mathbf{r}, \hat{\mathbf{p}}, t) \tag{2}$$

which can easily be solved for any initial-value problem $\psi(\mathbf{r}, p, \hat{\mathbf{p}}, t=0) = S_0(\mathbf{r}, p, \hat{\mathbf{p}})$ to get $\psi(\mathbf{r}, p, \hat{\mathbf{p}}, t) = e^{-t/\tau} S_0(\mathbf{r} - vt\hat{\mathbf{p}}, p, \hat{\mathbf{p}})$. This solution implies that the particles simply stream freely outward, all the while getting absorbed on a time scale τ. We shall see below just how poor the RTA is for the type of initial-value problems considered here.

Since Eq. (1) is very difficult to solve for an arbitrarily distributed source, some simplifying restrictions will be imposed. If plane symmetry is assumed, then the distribution function depends only on the coordinate normal to the plane x, the time t, and the direction cosine $\mu = \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}$; Eq. (1) then simplifies to

$$\frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} + \sigma\psi = \frac{\sigma}{2} \int_{-1}^{+1} d\mu' \, \psi(x, \mu', t) + S(x, \mu, t) \tag{3}$$

where units have been chosen so that $v = 1$.

The goal is to find exact expressions for the initial-value GFs, which, due to the linearity of the Boltzmann equation, immediately allow the distribution function and the density to be found for an arbitrary, plane symmetric source. The initial-value GF is defined to be the function $G_0(x, \mu, t; \mu_0)$ that solves Eq. (3) with a pulsed, monodirectional, planar source [i.e., $S(x, \mu, t) = \delta(x)\,\delta(t))\,\delta(\mu - \mu_0)$] and therefore satisfies the initial condition $G_0(x, \mu, t=0; \mu_0) = S_0(x, \mu) = \delta(x)\,\delta(\mu - \mu_0)$ and boundary conditions

$$\begin{aligned} G_0(x, \mu, t; \mu_0) &= 0, \qquad t < 0 \\ G_0(x, \mu, t; \mu_0) &= 0, \qquad |x| > t \end{aligned} \tag{4}$$

The density GF is defined as

$$\rho_0(x, t; \mu_0) \equiv \int d\hat{\Omega} \, G_0(x, \mu, t; \mu_0) = 2\pi \int_{-1}^{+1} d\mu \, G_0(x, \mu, t; \mu_0) \tag{5}$$

The angular GF for a planar, isotropic source $G_0(x, \mu, t)$ is defined as the solution to Eq. (3) with $S(x, \mu, t) = \frac{1}{2}\delta(x)\,\delta(t)$, and the corresponding density GF is

$$\rho(x, t) \equiv \int d\hat{\Omega}\; G_0(x, \mu, t) = 2\pi \int_{-1}^{+1} d\mu\; G_0(x, \mu, t) \tag{6}$$

Clearly

$$G_0(x, \mu, t) = \tfrac{1}{2} \int_{-1}^{+1} d\mu'\; G_0(x, \mu, t; \mu')$$

and

$$\rho(x, t) = \tfrac{1}{2} \int_{-1}^{+1} d\mu'\; \rho_0(x, t; \mu')$$

and the Fourier- and Laplace-transformed versions of these relations will be useful below in deriving the exact solutions.

The details of the derivation of $G_0(x, \mu, t; \mu_0)$ are given below, but only the final results for the other GFs are displayed, since the derivations for these are similar to the ones for $G_0(x, \mu, t; \mu_0)$ and $\rho(x, t)$ (the last given in ref. 21).

Following the procedure in ref. 21, the Boltzmann equation is Fourier transformed with respect to the spatial coordinate x and Laplace transformed with respect to the time t, giving

$$[s - ik\mu + \sigma]\; \tilde{G}_0(k, \mu, s; \mu_0) = \frac{\sigma}{2} \int_{-1}^{+1} d\mu'\; \tilde{G}_0(k, \mu', s; \mu_0) + \delta(\mu - \mu_0) \tag{7}$$

where

$$\tilde{G}_0(k, \mu, s; \mu_0) = \int_0^{\infty} dt\, e^{-st} \int_{-\infty}^{+\infty} dx\, e^{ikx} G_0(x, \mu, t; \mu_0) \tag{8}$$

(s is a complex number with positive real part and k is real). It is assumed that the integral in Eq. (8) exists, and therefore the order in which the two integral transforms are applied is unimportant [it will be implicitly understood throughout that $|x| \leqslant t$; see Eq. (4)].

Formally, $G_0(x, \mu, t; \mu_0)$ is found by inverting the two integral transforms:

$$G_0(x, \mu, t; \mu_0) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} ds\, e^{st} \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{-ikx} \tilde{G}_0(k, \mu, s; \mu_0)$$

where $a \in \mathbf{R}$ is to the right of all singularities of $\tilde{G}_0(k, \mu, s; \mu_0)$ in the complex s-plane, and the order of integration is unimportant.

Dividing Eq. (7) by $[s - ik\mu + \sigma]$ and then integrating the resulting equation over the whole solid angle leads to an expression for the double integral transform of $\rho_0(x, t; \mu_0)$,[(21),3]

$$\tilde{\rho}_0(k, s; \mu_0) = \frac{2\pi}{1 - (\sigma/k)\tan^{-1}[k/(s+\sigma)]} \frac{1}{s - ik\mu_0 + \sigma} \tag{9}$$

Plugging this back into Eq. (7) yields an expression for $\tilde{G}_0(k, \mu, s; \mu_0) = \tilde{G}_0^{(1)}(k, \mu, s; \mu_0) + \tilde{G}_{\text{unc}}(k, \mu, s; \mu_0)$, where

$$\begin{aligned} \tilde{G}_0^{(1)}(k, \mu, s; \mu_0) \\ &= \frac{\sigma \tilde{\rho}_0(k, s; \mu_0)}{4\pi[s - ik\mu + \sigma]} && (10) \\ &= \frac{\sigma}{2} \frac{1}{\{1 - (\sigma/k)\tan^{-1}[k/(s+\sigma)]\}[s - ik\mu_0 + \sigma][s - ik\mu + \sigma]} && (11) \end{aligned}$$

and

$$\tilde{G}_{\text{unc}}(k, \mu, s; \mu_0) = \frac{\delta(\mu - \mu_0)}{s - ik\mu_0 + \sigma} \tag{12}$$

is the contribution from the uncollided particles, which can immediately be inverted to find

$$G_{\text{unc}}(x, \mu, t; \mu_0) = (1/t)\, e^{-\sigma t} \delta(\mu - x/t)\, \delta(\mu - \mu_0) \tag{13}$$

The Laplace inversion of $\tilde{G}_0^{(1)}(k, \mu, s; \mu_0)$,

$$\tilde{G}_0^{(1)}(k, \mu, \mu_0) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} ds\, e^{st} \tilde{G}_0^{(1)}(k, \mu, s; \mu_0) \tag{14}$$

can be performed by noting that in the complex s-plane $\tilde{G}_0^{(1)}(k, \mu, s; \mu_0)$ has (i) a simple isolated *hydrodynamic* pole at $s_0(k) = k\cot(k/\sigma) - \sigma$, which exists only for $|k| \leqslant \pi\sigma/2$, (ii) a branch cut along the line joining $s_+ = -\sigma + ik$ and $s_- = -\sigma - ik$, and (iii) two poles embedded in the cut at $s_1 = -\sigma + ik\mu_0$ and $s_2 = -\sigma + ik\mu$ (see Fig. 1).[(21)] We see that $\tilde{G}_0^{(1)}(k, \mu, s; \mu_0)$ arises entirely from scattering processes and Eq. (10) shows that the local density $\tilde{\rho}_0(k, s; \mu_0)$ acts as an effective source of particles that can be scattered into the state in question. It should be kept in mind that

[3] The factor of 1/2 in Eq. (15) of ref. 21 should be a 2.

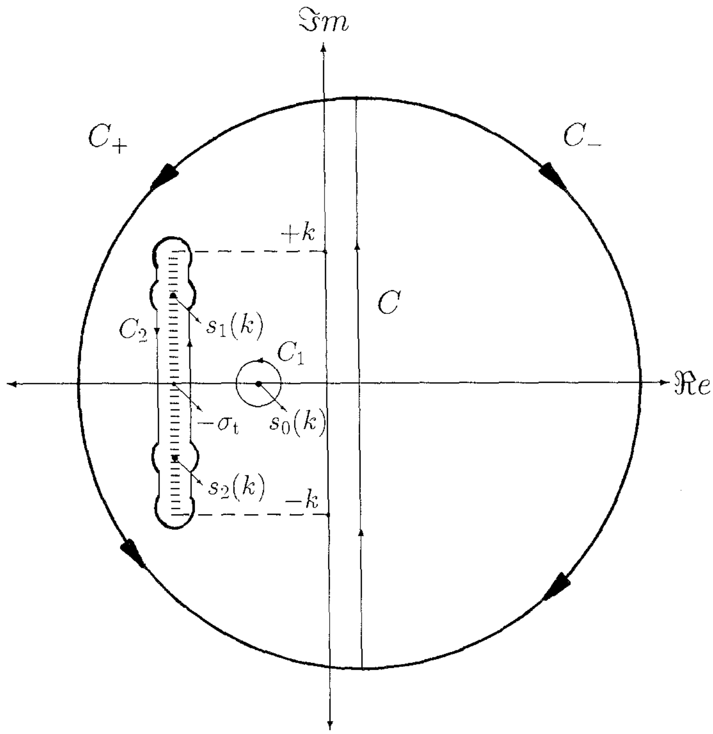

Fig. 1. The complex s-plane, showing the singularity structure of the GF $\tilde{G}_0(k, \mu, s; \mu_0)$ and the integration contours used in the Laplace inversion of the GFs.

for the present problem without scattering only the states with $\mu = \mu_0$ would be occupied. The complicated analytic structure of $\tilde{G}_0^{(1)}(k, \mu, s; \mu_0)$ reflects the various routes by which the particles initially (at $t = 0$) in the state $\mu = \mu_0$ can reach and be scattered out of the state μ at the space-time point (x, t). For example, Eq. (13) shows how the initial disturbance gets damped as particles undergo collisions and we see how this damping arises from the pole at s_1 in Eq. (12). As I will discuss in greater detail below, the poles at s_1 and s_2 in $\tilde{G}_0^{(1)}$ describe, respectively, the depopulation of the state in question through the scattering out of the state and the population of the state through the direct scattering of particles from μ_0 to μ. The branch cut describes the population of the state in question through the scattering of the original particles in the state μ_0 through an intermediate state μ' to the final state μ at the space-time point (x, t). The hydrodynamic pole can roughly be thought of as providing the contribution to the state in question from the sum of all processes containing more than one intermediate state. As we will see below, this term contains the usual diffusive relaxation behavior expected to occur at long times and long wavelengths.

By closing the Bromwich contour in the left half-plane and then deforming the closed contour, the Laplace inversion for $t>0$ reduces to the evaluation of two contour integrals, one around the isolated pole and the other around the cut, implying that $\tilde{G}_0^{(1)}(k, \mu, t; \mu_0)$ breaks up into two pieces, $\tilde{G}_0^{(1)}(k, \mu, t; \mu_0) = \tilde{G}_{\text{pole}}(k, \mu, t; \mu_0) + \tilde{G}_{\text{bc}}(k, \mu, t; \mu_0)$.

The pole contribution is

$$\tilde{G}_{\text{pole}}(k, \mu, t; \mu_0) = \frac{1}{2\pi i} \int_{C_1} ds\, e^{st} \tilde{G}_0^{(1)}(k, \mu, s; \mu_0) \tag{15}$$

which is easily evaluated using Eq. (11) and the residue theorem (C_1 is the contour around the pole, see Fig. 1); performing the Fourier inversion then yields, after some manipulation,

$$\begin{aligned} &G_{\text{pole}}(x, \mu, t; \mu_0) \\ &\quad = \frac{1}{2\pi} \int_0^{+\pi\sigma/2} dk \exp\left\{ -t \left[\sigma - k \cot\left(\frac{k}{\sigma}\right) \right] \right\} \\ &\qquad \times \csc^2\left(\frac{k}{\sigma}\right) \frac{\cos(kx)[\cot(k/\sigma) - \mu\mu_0] + \sin(kx)(\mu + \mu_0)\cot(k/\sigma)}{[\cot^2(k/\sigma) - \mu\mu_0)]^2 + \cot^2(k/\sigma)(\mu + \mu_0)^2} \end{aligned} \tag{16}$$

The pole piece is clearly nonpropagating (diffusive rather than ballistic), which justifies the name *hydrodynamic pole*; it is also the asymptotic solution, since its decay rate $\sigma - k\cot(k/\sigma)$ [which goes as $(1/3\sigma)\, k^2$ for small $k \ll \sigma$] is slower than the decay rate σ of the branch cut (see below) and uncollided pieces, which are the transport transient (propagating) parts of the solution. Because only modes with wavenumber $|k| \leqslant \pi\sigma/2$ contribute to the expression for $G_{\text{pole}}(x, \mu, t; \mu_0)$, the hydrodynamic pole piece contains no information about length scales $\lambda = 2\pi/k < 4l$, where $l \equiv 1/\sigma$ (recall that $v=1$) is the mean free path (MFP) for particle scattering. In fact, $G_{\text{pole}}(x, \mu, t; \mu_0)$ is the exact Chapman–Enskog solution (cf. ref. 8) for the distribution function. Since for the present problem the actual initial distribution does not lie in the *hydrodynamic functional subspace*, the Chapman–Enskog solution is not the whole story: If we define a projection operator by

$$\hat{P} \equiv \int \frac{d\hat{\Omega}}{4\pi}$$

and a function $W(k, \mu) \equiv \sigma[k\cot(k/\sigma) - ik\mu]^{-1}$, then a (Fourier-transformed) initial distribution $\tilde{\psi}(k, \mu, t=0) = \tilde{S}_0(k, \mu)$ lies in the hydrodynamic subspace iff the following identity is satisfied[8]:

$$\tilde{S}_0(k, \mu) = \begin{cases} W\hat{P}(\tilde{S}_0) & \text{if } k \leqslant \pi\sigma/2 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

If the initial distribution satisfies this identity, then the Chapman–Enskog solution is the exact solution to the BE. If the initial distribution does not satisfy this identity, then the hydrodynamic pole solution is the exact solution of a generalized hydrodynamic equation (and not the BE) containing terms to all orders in the gradient operator (see below)

$$[s - s_0(k)]\, \tilde{\psi}_{\text{pole}}(k, \mu, s) = \tilde{S}'_0$$

but with modified (not the actual) initial condition $\tilde{S}'_0$ given by (in k space)

$$\tilde{S}'_0(k, \mu) = \begin{cases} W(k, \mu) \left(\dfrac{k}{\sigma}\right)^2 \csc^2\left(\dfrac{k}{\sigma}\right) \hat{P}(W\tilde{S}_0) & \text{if } \ k \leqslant \dfrac{\pi\sigma}{2} \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

which is the projection of the actual initial distribution $\tilde{S}_0(k, \mu)$ into the hydrodynamic functional subspace. The hydrodynamic pole solution is the part of the exact distribution function that can be obtained by summing a perturbation series in powers of $l/\lambda \sim \tau/T$, where λ is the characteristic length scale and T the characteristic time scale of the disturbance. As we will see below, the part of the distribution function coming from $\tilde{G}_0^{(1)}(k, \mu, s; \mu_0)$ is nonperturbative in these expansion parameters. The expression for $G_{\text{pole}}(x, \mu, t; \mu_0)$ can easily be numerically evaluated, but these results will not be presented here.

The branch cut contribution to $\tilde{G}_0^{(1)}(k, \mu, s; \mu_0)$ is

$$\tilde{G}_{\text{bc}}(k, \mu, t; \mu_0) = \frac{1}{2\pi i} \int_{C_2} ds\, e^{st} \tilde{G}_0^{(1)}(k, \mu, s; \mu_0)$$

where C_2 is the contour around the branch cut and the embedded poles (see Fig. 1). The evaluation of this contour integral can be slightly simplified by using the identity

$$\frac{1}{[s - ik\mu_0 + \sigma][s - ik\mu + \sigma]} = \frac{1}{(ik)(\mu - \mu_0)} \left\{ \frac{1}{s - ik\mu + \sigma} - \frac{1}{s - ik\mu_0 + \sigma} \right\}$$

If

$$\tilde{F}(k, \mu, s) \equiv \frac{1}{ik} [s - ik\mu + \sigma]^{-1} \left[1 - \frac{\sigma}{k} \tan^{-1}\left(\frac{k}{s + \sigma}\right) \right]^{-1} \tag{19}$$

then

$$\tilde{G}_0^{(1)}(k, \mu, s; \mu_0) = \frac{\sigma}{2(\mu - \mu_0)} [\tilde{F}(k, \mu, s) - \tilde{F}(k, \mu_0, s)] \tag{20}$$

Due to the presence of the embedded pole, the Laplace inversion of $\tilde{F}(k, \mu, s)$

$$\tilde{F}(k, \mu, t) = \frac{1}{2\pi i} \int_{C_2} ds\, e^{st} \tilde{F}(k, \mu, s)$$

must be performed with care (see ref. 14 for the details of how to handle this situation). The answer is

$$\tilde{F}(k, \mu, t) = -\frac{\sigma}{2\pi} e^{-\sigma t} \left\{ \mathscr{P} \int_{-1}^{+1} d\alpha \frac{e^{ik\alpha t}}{[\alpha - \mu]} \frac{1}{[k - z_1(\alpha)][k - z_2(\alpha)]} \right.$$
$$\left. + 2\pi i e^{ik\mu t} \frac{k + i(\sigma/2) \ln |(1+\mu)/(1-\mu)|}{[k - z_1(\mu)][k - z_2(\mu)]} \right\} \tag{21}$$

where the first term comes from the line integral of the discontinuity of $\tilde{F}(k, \mu, s)$ along the cut, and the second from the semicircular parts of the contour around the embedded pole at $s = -\sigma + ik\mu$ ($\mathscr{P}$ denotes Cauchy principal value). The discontinuity of the function

$$\left[1 - \frac{\sigma}{k} \tan^{-1} \left(\frac{k}{s + \sigma} \right) \right]^{-1}$$

across the cut at $s(\alpha) = -\sigma + ik\alpha$ ($|\alpha| \leqslant 1$) is

$$\frac{\sigma k}{[k - z_1(\alpha)][k - z_2(\alpha)]}$$

where

$$z_1(\alpha), z_2(\alpha) \equiv \frac{\sigma}{2} \left\{ \pm \pi - i \ln \left| \frac{1+\alpha}{1-\alpha} \right| \right\}$$
$$= \frac{\sigma}{2} \{ \pm \pi - 2i \tanh^{-1}(\alpha) \} \tag{22}$$

and this has been used in the derivation of Eq. (21).

The Fourier inversion of the first piece of $\tilde{F}(k, \mu, t)$ can now be done by interchanging the α and Fourier integrals (see ref. 21 for a discussion of the validity of this operation) and then making use of the Cauchy residue theorem. The Fourier inversion of the second piece of $\tilde{F}(k, \mu, t)$ can be obtained by applying the residue theorem directly. The original contour along the real axis can be closed in either the upper or lower half of the complex k-plane (depending on the sign of x; for details see ref. 21). Then it is a simple matter to pick up the contribution of the simple poles at z_1 and z_2 in Eq. (21) (see Fig. 2), with the result

$$F(x, \mu, t) = \operatorname{sgn}(x) \frac{2}{\pi} e^{-\sigma t} \mathscr{P} \int_0^{|x|/t} \frac{d\alpha}{\alpha - \operatorname{sgn}(x)\, \mu} \left| \frac{1-\alpha}{1+\alpha} \right|^{\sigma(|x| - \alpha t)/2} \times \sin\left(\sigma\pi \frac{|x| - \alpha t}{2} \right) + e^{-\sigma t} J(x, \mu, t) \tag{23}$$

where

$$J(x, \mu, t) \equiv \operatorname{sgn}(x) \left| \frac{1-\mu}{1+\mu} \right|^{\sigma(x - \mu t)/2} \cos\left[\frac{\sigma\pi(x - \mu t)}{2} \right] \tag{24}$$

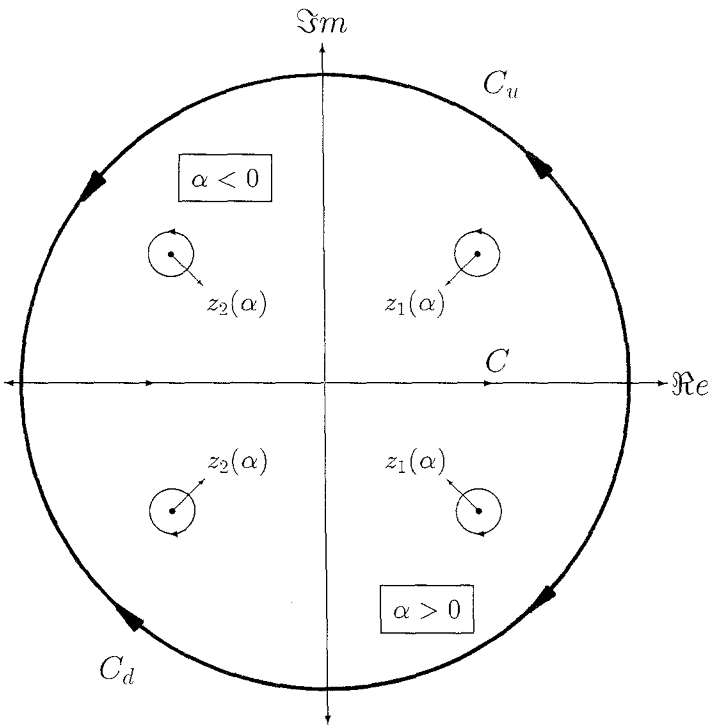

Fig. 2. The complex k-plane, showing the poles and integration contours used in the Fourier inversion of the function $\tilde{F}(k, \mu, t)$.

if $0 \leqslant \mu \leqslant x/t \leqslant 1$ or $-1 \leqslant x/t \leqslant \mu \leqslant 0$, and zero otherwise (since some of the functions considered here tend to be somewhat singular, there are some delicate points connected with the Fourier inversions; see ref. 21 for details).

Then Eq. (20) implies that

$$
\begin{aligned}
&G_{\mathrm{bc}}(x, \mu, t; \mu_0) \\
&\quad = \frac{\sigma}{2(\mu - \mu_0)} \left[F(x, \mu, t) - F(x, \mu_0, t) \right] \\
&\quad = \operatorname{sgn}(x) \frac{\sigma}{\pi} e^{-\sigma t} \mathscr{P} \int_0^{|x|/t} d\alpha \left| \frac{1-\alpha}{1+\alpha} \right|^{\sigma(|x| - \alpha t)/2} \frac{\sin[\sigma\pi(|x| - \alpha t)/2]}{[\alpha - \operatorname{sgn}(x)\, \mu][\alpha - \operatorname{sgn}(x)\, \mu_0]} \\
&\qquad + \frac{\sigma e^{-\sigma t}}{2(\mu - \mu_0)} \left[J(x, \mu, t) - J(x, \mu_0, t) \right] \qquad (25)
\end{aligned}
$$

A few comments are in order: The branch cut contribution $G_{\mathrm{bc}}(x, \mu, t; \mu_0)$ is a sum of damped ($\propto e^{-\sigma t}$) traveling [depending on $(x - \alpha t)$] sine waves all with the same wavelength $\lambda = 4/\sigma$, but with different velocities (0 to $v = 1$) in the x direction. $G_{\mathrm{bc}}(x, \mu, t; \mu_0)$ and $G_{\mathrm{unc}}(x, \mu, t; \mu_0)$ make up the transport transient part of the solution, decaying in time according to $e^{-\sigma t}$, and, roughly speaking, these terms account for the contribution to the distribution function from those particles that, although they may or may not have undergone some collisions, are still propagating ballistically over distances $\sim vt$; $G_{\mathrm{unc}}(x, \mu, t; \mu_0)$ is the contribution of a subclass of these particles, namely those that have not experienced any collisions at all. On the other hand, the pole contribution comes from those particles that have undergone many collisions, and at times $t \gg \tau$ the bulk of the particles are in this class. The $e^{-\sigma t} = \exp\{-1/(\tau/t)\}$ dependence of the transport transient part of the distribution function precludes a power series expansion in τ/t, since such a term has an essential singularity at $\tau/t = 0$, and we clearly see that the transport-transient, propagating parts of the distribution are nonperturbative in nature.

The final result for $G_0(x, \mu, t; \mu_0)$ is

$$
\begin{aligned}
G_0(x, \mu, t; \mu_0) = [&G_{\mathrm{unc}}(x, \mu, t; \mu_0) + G_{\mathrm{bc}}(x, \mu, t; \mu_0) \\
&+ G_{\mathrm{pole}}(x, \mu, t; \mu_0)]\, \Theta(1 - |x|/t) \qquad (26)
\end{aligned}
$$

with $G_{\mathrm{unc}}(x, \mu, t; \mu_0)$, $G_{\mathrm{bc}}(x, \mu, t; \mu_0)$, and $G_{\mathrm{pole}}(x, \mu, t; \mu_0)$ given by Eqs. (13), (25), and (16) (the step function enforces the boundary condition). In principle, these expressions can be numerically evaluated, but these results will not be presented here. Using a completely different method (summing the multiple collision expansion), Ganapol[14] has derived a much less transparent expression for $G_0(x, \mu, t; \mu_0)$ in the form of a contour integral, but after some work one can show that his result reduces to the simpler form obtained above.[22]

Before moving on, I would like to discuss the physical interpretation of the various pieces appearing in the final exact solution (26) (cf. Section 9 of ref. 18). In the apparent complexity of the final solution it is important to keep in mind that the basic physics is simple: particles are propagating ballistically with a speed $v = 1$ until they collide and scatter on a time scale τ. Thus, the initial distribution of particles with $\mu = \mu_0$ is depopulated with the decay products appearing at other μ, and the particles in these newly populated states in turn propagate and decay. The complexity of the final solution comes from the fact that starting with an initial distribution of particles in state μ_0 at $x = 0$, there are many different routes for a particle to appear in state μ at the space-time point (x, t). The function $G_{\text{unc}}(x, \mu, t; \mu_0)$ describes the particles that have not undergone any collisions, and all the particles in this class can be traced directly back to the initial distribution of particles in the state $\mu = \mu_0$ at $t = 0$. The uncollided piece decays as $e^{-\sigma t}$, which is physically sensible, since the initial distribution of particles is undergoing collisions on the time scale $\tau = 1/\sigma$. The second, nonintegral piece of $G_{\text{bc}}(x, \mu, t; \mu_0)$ [Eq. (25)] comes from the two poles s_1 and s_2 embedded in the branch cut (see Fig. 1) and contains two pieces: one proportional to the function J evaluated at μ (the state in question) and the second proportional to the function J evaluated at μ_0 (the state of the initial distribution). We interpret the first part $[\propto J(x, \mu, t)]$ as describing the *depopulation* of the state in question due to scattering out of this state and the second part $[\propto J(x, \mu_0, t)]$ as describing the *population* of the state μ due to the particles that are being scattered directly from μ_0 to μ. The integral term in $G_{\text{bc}}(x, \mu, t; \mu_0)$ describes the filling of the state μ by particles that have scattered from the initial state μ_0 to an intermediate state μ' and then finally to the state μ. Since any state μ' can serve as an intermediate state, there is naturally an integral over all possibilities. The hydrodynamic pole piece can then be interpreted as the contribution to the distribution function from the sum of all the contributions arising from all the higher order processes where the particles in the initial state μ_0 reach the final state μ by passing through more than one intermediate state. The part of the distribution function arising from this infinite sum contains a qualitatively new, diffusive rather than ballistic, behavior. In the late-time $t \gg \tau$ and long-wavelength $\lambda \gg l$ limit, the pole piece dominates the particle distribution function as particles no longer can transmit their influence by direct ballistic propagation from one region of the system to another separated by more than a few MFPs; instead, the particles collide many times locally, executing a sort of random walk through the matrix of scattering centers. I will further discuss this interpretation below.

In a similar way, the transform inversion of $\tilde{\rho}_0(k, s; \mu_0)$ [Eq. (9)] can be performed, resulting in

$$\rho_0(x, t; \mu_0) = [\rho_{\text{unc}}(x, t; \mu_0) + \rho_{\text{bc}}(x, t; \mu_0) + \rho_{\text{pole}}(x, t; \mu_0)]\, \Theta(1 - |x|/t) \tag{27}$$

where

$$\rho_{\text{unc}}(x, t; \mu_0) = 2\pi e^{-\sigma t} \delta(\mu_0 t - x) \tag{28}$$

is the uncollided piece,

$$\rho_{\text{bc}}(x, t; \mu_0) = \rho_1(x, t; \mu_0) + \rho_2(x, t; \mu_0) \tag{29}$$

comes from the branch cut contribution, with

$$\rho_1(x, t; \mu_0) = -\sigma e^{-\sigma t} \mathscr{P} \int_0^{|x|/t} \frac{d\alpha}{\alpha - \operatorname{sgn}(x)\, \mu_0} \left| \frac{1-\alpha}{1+\alpha} \right|^{\sigma(|x| - \alpha t)/2} \times \left\{ \pi \cos\left[\frac{\sigma\pi(|x| - \alpha t)}{2} \right] - \ln\left| \frac{1+\alpha}{1-\alpha} \right| \sin\left[\frac{\sigma\pi(|x| - \alpha t)}{2} \right] \right\} \tag{30}$$

and

$$\rho_2(x, t; \mu_0) = -2\pi \operatorname{sgn}(x)\, e^{-\sigma t} \left| \frac{1-\mu_0}{1+\mu_0} \right|^{\sigma(x - \mu_0 t)/2} \times \left\{ \ln\left| \frac{1+\mu_0}{1-\mu_0} \right| \cos\left[\frac{\sigma\pi(x - \mu_0 t)}{2} \right] + \pi \sin\left[\frac{\sigma\pi(x - \mu_0 t)}{2} \right] \right\} \tag{31}$$

if $0 \leqslant \mu_0 \leqslant x/t \leqslant 1$ or $-1 \leqslant x/t \leqslant \mu_0 \leqslant 0$, and zero otherwise; and finally the pole contribution is

$$\rho_{\text{pole}}(x, t; \mu_0) = 2 \int_0^{+\pi\sigma/2} dk \exp\left\{ -t \left[\sigma - k \cot\left(\frac{k}{\sigma} \right) \right] \right\} \times \csc^2\left(\frac{k}{\sigma} \right) \left(\frac{k}{\sigma} \right) \frac{\cos(kx) \cot(k/\sigma) + \mu_0 \sin(kx)}{\cot^2(k/\sigma) + \mu_0^2} \tag{32}$$

which is the exact Chapman–Enskog expression for the density.

Integrating $\tilde{G}_0(k, \mu, s; \mu_0)$ over μ_0 [see Eqs. (11) and (12)] leads to an expression for the double integral transform of the isotropic source GF $G_0(x, \mu, t)$,

$$\tilde{G}_0(k, \mu, t) = \frac{\sigma}{2k} \frac{\tan^{-1}[k/(s+\sigma)]}{1 - (\sigma/k)\tan^{-1}[k/(s+\sigma)]} \frac{1}{s - ik\mu + \sigma} + \frac{1}{2[s - ik\mu + \sigma]}$$

But then a little algebra shows that this expression is just $(1/4\pi)\,\tilde{\rho}_0(k, s; \mu)$ [see Eq. (9)], and therefore

$$G_0(x, \mu, t) \equiv \frac{1}{4\pi}\,\rho_0(x, t; \mu)$$

which comes from integrating the reciprocity relation $G_0(x, \mu, t; \mu_0) = G_0(x, \mu_0, t; \mu)$ over μ_0.

Finally, the expression for $\rho(x, t)$ (already obtained in ref. 21) is

$$\rho(x, t) = [\rho_{\text{unc}}(x, t) + \rho_1(x, t) + \rho_{\text{pole}}(x, t)]\, \Theta(1 - |x|/t)$$

where

$$\rho_{\text{unc}}(x, t) = \frac{\pi}{t} e^{-\sigma t} \tag{33}$$

is the uncollided part,

$$\rho_{\text{pole}}(x, t) = 2e^{-\sigma t} \int_0^{+\pi\sigma/2} dk \exp\{kt \cot(k/\sigma)\} \csc^2(k/\sigma)(k/\sigma)^2 \cos(kx) \tag{34}$$

is the *hydrodynamic* pole piece and the exact Chapman–Enskog expression for the density, and

$$\rho_1(x, t) = -\frac{\sigma}{2} e^{-\sigma t} \int_0^{|x|/t} d\alpha \left|\frac{1-\alpha}{1+\alpha}\right|^{\sigma(|x| - \alpha t)/2} \left\{2\pi \ln\left|\frac{1+\alpha}{1-\alpha}\right| \cos\left[\frac{\sigma\pi(|x| - \alpha t)}{2}\right] + \left(\pi^2 - \ln^2\left|\frac{1+\alpha}{1-\alpha}\right|\right) \sin\left[\frac{\sigma\pi(|x| - \alpha t)}{2}\right]\right\} \tag{35}$$

is the branch cut contribution (with the uncollided part subtracted out).

The part $\rho_{\text{pole}}(x, t)$ is the exact solution of the generalized hydrodynamic equation (cf. ref. 8),

$$\left[\frac{\partial}{\partial t} + 2\sum_{m=1}^{\infty} \frac{B_m}{(2m)!}\left(\frac{2}{\sigma}\right)^{2m-1}\left(-\frac{\partial^2}{\partial x^2}\right)^m\right] n(x, t) = 0 \tag{36}$$

but with a modified initial condition, not the actual one $\rho(x, 0) = 2\pi\delta(x)$

(the B_m are the Bernoulli numbers). The generalized hydrodynamic equation is obviously an expansion in the MFP $l \propto 1/\sigma$, so that if the MFP is small compared with the characteristic length scale of the disturbance, then one might be able to get away with keeping only a few terms in the sum, but even by keeping terms to all orders in the spatial derivatives the solution will completely miss the nonperturbative, propagating modes coming from $\rho_{\rm unc}(x, t)$ and $\rho_1(x, t)$.

It is interesting to see explicitly how the generalized hydrodynamic equation arises for this specific problem (cf. ref. 8). The idea is to find a function having only a simple pole, which, when transformed back to (x, t) space, reproduces $\rho_{\rm pole}(x, t)$. The right function is

$$\tilde{n}(k, s) = \frac{2\pi \csc^2(k/\sigma)(k/\sigma)^2}{s - H(k)} \tag{37}$$

where $H(k) \equiv -\sigma + k \cot(k/\sigma)$. Then by Fourier and Laplace transforming $[s - H(k)] \times$ Eq. (37), it is easy to show that $n(x, t)$ satisfies the generalized hydrodynamic equation with initial condition $n(x, 0) = \mathscr{F}^{-1}[\tilde{n}_0(k)]$, where $\mathscr{F}^{-1}$ denotes the Fourier inversion operator and $\tilde{n}_0(k) \equiv 2\pi \csc^2(k/\sigma)(k/\sigma)^2$ for $|k| \leqslant \pi\sigma/2$ and 0 otherwise. The simple diffusion theory (*Navier–Stokes*) result can be recovered by expanding all quantities to lowest nonvanishing order in k (or lowest order in the spatial derivatives) in Eq. (37) and reverting to (x, t) space (cf. ref. 16):

$$\left[\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right] n(x, t) = 0 \tag{38}$$

where $D = 1/3\sigma$ is the diffusion constant. As shown in ref. 8, at the Navier–Stokes level of description the hydrodynamic moment of the initial distribution *can* be used as the initial condition for solving the hydrodynamic equation, since the corrections appear only at higher orders in an expansion in $k/\sigma \sim l/\lambda$ (this simplification breaks down at the next level of description, that of the Burnett equation). With this simplification in mind we can solve the simple diffusion equation (38) with the hydrodynamic moment of the actual initial distribution as appropriate initial data to find the usual diffusion theory result,

$$n(x, t) \sim (4\pi D t)^{-1/2}\, e^{-x^2/4Dt} \tag{39}$$

In general, however, we see that due to the coarse graining, the hydrodynamic moment of the initial distribution function $\rho(x, 0) = 2\pi\delta(x)$ is *not* the correct initial data for the contracted (generalized hydrodynamic) description (36) (cf. ref. 8). We reemphasize, however, that if the initial distribution lies in the hydrodynamic subspace (see earlier discussion), then the density obtained from the generalized hydrodynamic equation (with the hydrodynamic moment of the initial distribution used as initial conditions) is the *exact* solution for the density as would be obtained from the exact solution to the BE. I postpone a further analysis of the exact solution for the density until the next section, where I discuss the decay of localized disturbances in a spherical geometry.

The full expression for $\rho(x, t)$ can be evaluated numerically, and where they can be compared the results are in perfect agreement[22] (to within the stated accuracy) with the numerical results obtained (in a completely different way) using the multiple collision method.[15,16] Presentation of numerical results, however, will be restricted to the more physically interesting spherical symmetry problems discussed in the next section.

3. SPHERICAL SYMMETRY TRANSPORT PROBLEMS

Although the plane symmetry problem can be completely solved (Section 2), it is not as physically interesting as the more difficult spherical symmetry problem, which is closer to the situation envisaged in the Introduction: a pulsed disturbance initially localized in some small region of a large system.

It is not easy to adapt the standard solution techniques to solve the spherical symmetry problem, but a density GF can immediately be obtained from the plane symmetry density GF [now denoted by $\rho_{\text{pl}}(x, t)$] derived in the previous section.[21]

The distribution function due to an isotropic, but otherwise arbitrary, spherically symmetric source is given by the solution to the spherically symmetric version of Eq. (1):

$$\frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial \psi}{\partial \mu} + \sigma \psi = \frac{\sigma}{2} \int_{-1}^{+1} d\mu' \, \psi(r, \mu', t) + S(r, t) \tag{40}$$

where r is the radius, and μ is now defined to be the direction cosine of the velocity with the position vector.

If ρ_{ss} denotes the density due to an isotropic, pulsed shell source located at $r = r_0$ and triggered at $t = t_0$ [i.e., $S(r, t) = (1/4\pi r_0^2)\, \delta(r - r_0)$ $\delta(t - t_0)$], then, as discussed in ref. 21,

$$\rho_{\text{ss}}(r, t; r_0, t_0) = \frac{1}{4\pi r_0 r} \left[\rho_{\text{pl}}(|r - r_0|, t - t_0) - \rho_{\text{pl}}(r + r_0, t - t_0)\right] \tag{41}$$

and the linearity of the Boltzmann equation implies that the density $\phi(r, t)$ due to an arbitrary isotropic source S is given by

$$\phi(r, t) = 4\pi \int dr_0\, r_0^2 \int dt_0\, \rho_{ss}(r, t; r_0, t_0)\, S(r_0, t_0) \tag{42}$$

In particular, the density due to an isotropic, pulsed point source [i.e., $S(r, t) = (1/4\pi r^2)\,\delta(r)\,\delta(t)$] can be found from $\rho_{pl}(x, t)$ using

$$\rho_{pt}(r, t) = -\frac{1}{2\pi r}\left(\frac{\partial \rho_{pl}(x, t)}{\partial x}\right)_{x=r} \tag{43}$$

which can be evaluated, yielding the following closed form expression:

$$\rho_{pt}(r, t) = [\rho^{pt}_{unc}(r, t) + \rho^{pt}_{1}(r, t) + \rho^{pt}_{bc}(r, t) + \rho^{pt}_{pole}(r, t)]\ \Theta(1 - r/t) \tag{44}$$

where

$$\rho^{pt}_{unc}(r, t) = \frac{1}{4\pi r t}\, e^{-\sigma t}\delta(r - t) \tag{45}$$

is the *uncollided* contribution,

$$\rho^{pt}_{1}(r, t) = \frac{\sigma}{2rt}\, e^{-\sigma t} \ln\left|\frac{t + r}{t - r}\right| \tag{46}$$

is the first collided piece (due to the particles that have undergone exactly one collision),

$$\begin{aligned}\rho^{pt}_{bc}(r, t) = \frac{\sigma^2}{8\pi r}\, e^{-\sigma t} \int_0^{r/t} d\alpha \left|\frac{1-\alpha}{1+\alpha}\right|^{\sigma(r-\alpha t)/2} &\left\{\left(\pi^3 - 3\pi \ln^2\left|\frac{1+\alpha}{1-\alpha}\right|\right)\right. \\ &\left.\times \cos\left[\frac{\sigma\pi(r-\alpha t)}{2}\right] - \ln\left|\frac{1+\alpha}{1-\alpha}\right|\left(3\pi^2 - \ln^2\left|\frac{1+\alpha}{1-\alpha}\right|\right)\sin\left[\frac{\sigma\pi(r-\alpha t)}{2}\right]\right\}\end{aligned} \tag{47}$$

is due to the branch cut, and finally

$$\rho^{pt}_{pole}(r, t) = (\sigma/\pi r)\int_0^{+\pi\sigma/2} dk\, \exp\{-t[\sigma - \cot(k/\sigma)]\}\, \csc^2(k/\sigma)(k/\sigma)^3 \sin(kr) \tag{48}$$

is the hydrodynamic pole contribution.

Before presenting some numerical results based on the above exact solutions, I will first make a few comments. With the exact results for the density due to a point in hand, we are in a position to understand what is happening physically. The exact solutions to the isotropic, pulsed point source problem consists of four distinct parts: $\rho^{\mathrm{pt}}_{\mathrm{unc}}(r, t)$, $\rho^{\mathrm{pt}}_{1}(r, t)$, and $\rho^{\mathrm{pt}}_{\mathrm{bc}}(r, t)$ are the transport transient parts of the solution, decaying rapidly on the time scale τ, so that at late times only the slowly decaying, hydrodynamic pole part of the solution $\rho^{\mathrm{pt}}_{\mathrm{pole}}(r, t)$ contributes significantly to the density. Again the basic physics is the same as discussed earlier: The original particles in the point burst propagate out while undergoing collision on the time scale τ, and the products of the collisions appear in other states. The part $\rho^{\mathrm{pt}}_{\mathrm{unc}}(r, t)$ accounts for the part of the density due to those particles that have not yet undergone any collisions. In the absence of collisions there would be a delta-function shell propagating out with speed $v = 1$, and with collisions there is still such a shell, but it is being depopulated as particles collide. The part $\rho^{\mathrm{pt}}_{1}(r, t)$ describes the part of the density due to those particles that have undergone exactly one collision, and this piece contains a weak logarithmic singularity right at the wavefront, which reflects the fact that these *first collided* particles come from the decay of the initial delta function shell. Clearly, $\rho^{\mathrm{pt}}_{\mathrm{bc}}(r, t)$ and $\rho^{\mathrm{pt}}_{\mathrm{pole}}(r, t)$ must come from the sum over all terms containing the contributions from the particles that have undergone more than one collision. Again I emphasize that the complexity of these two pieces comes from the existence of many different routes by which a *multiply collided* particle can come to contribute to the density at the space-time point (r, t). This infinite sum contains a diffusive, or hydrodynamic, piece $\rho^{\mathrm{pt}}_{\mathrm{pole}}(r, t)$ that decays slowly and contains no traveling wave component; this is the exact Chapman–Enskog expression for the density. In the limit $r \gg l$ and $t \gg \tau$ this pole solution reduces to the simple diffusion theory result ρ_{diff} discussed below.

Some more insight into the pole solution can be obtained by recalling the following "derivation" of the simple diffusion equation from the BE.[1] If we expand the exact distribution function in spherical harmonics and truncate after the first two terms, we get

$$\psi(\mathbf{r}, \hat{\mathbf{p}}, t) = n(\mathbf{r}, t) + 3\hat{\mathbf{p}} \cdot \mathbf{J}(\mathbf{r}, t) \tag{49}$$

where

$$n(\mathbf{r}, t) = \int \frac{d\hat{\Omega}}{4\pi} \psi(\mathbf{r}, \hat{\mathbf{p}}, t)$$

is the density and

$$\mathbf{J}(\mathbf{r}, t) = \int \frac{d\hat{\Omega}}{4\pi} \hat{\mathbf{p}} \psi(\mathbf{r}, \hat{\mathbf{p}}, t)$$

is the current density. Then, by taking moments of the BE with respect to 1 and $\hat{\mathbf{p}}$, we obtain two coupled equations for n and $\mathbf{J}$, which in the limit $r \gg l$ and $t \gg \tau$ can be solved to find Fick's law $\mathbf{J} = -D\nabla n$ and the simple diffusion equation for the density n, which is the Navier–Stokes level of description. If we had instead kept only the first term in the spherical harmonic expansion of the distribution, we would have found the simple equation $\partial_t n(\mathbf{r}, t) = 0$, the Euler equation, which implies that at this (lowest) level of description *the density does not decay at all*, and this gives us a hint as to the origin of the slow decay in the exact pole solution.

With this prelude, we can get some more insight into the orgin of the slowly decaying diffusion mode described by $\rho_{\text{pole}}^{\text{pt}}(r, t)$. Within coarse-grained spatial regions of length scale $\sim l$, collisions very effectively smooth out large anisotropies (wrt $\hat{\mathbf{p}}$) in the distribution function, but neighboring regions will end up with slightly different densities and current densities. Thus, there will be a net flux of particles through the boundaries of these coarse-grained regions, and this will tend to decrease the density in the regions where there is a surplus and increase the density where there is a deficit. The smoothing out of the local anisotropies in the distribution function takes place on the rapid time scale τ, but the diffusive change in the density takes place only due to the slight imbalances from one coarse-grained region to another, and this diffusive relaxation takes places on a much slower time scale, which is in fact infinite for the $k = 0$ (homogeneous mode). This diffusive transport should be contrasted with ballistic transport contained in the other three terms, $\rho_{\text{unc}}^{\text{pt}}(r, t)$, $\rho_1^{\text{pt}}(r, t)$, and $\rho_{\text{bc}}^{\text{pt}}(r, t)$, which decay on the fast time scale τ and come from the highly anisotropic, nearly ballistic, part of the distribution function. (The free streaming of the initial point burst of particles is highly anisotropic, since all the particles are traveling along their radius vectors.) Although the basic qualitative ideas—connected with the existence of transport transients and asymptotic solutions—exemplified by the above exact results are well known in transport theory, the Lorentz model calculations allow us to replace suggestive handwaving with concrete results.

The above expressions can easily be numerically evaluated, and where they can be compared, the results for $\rho_{\text{pt}}(r, t)$ are in perfect agreement (to within the stated accuracy) with the results obtained using the multiple collision method.[16,22] One of the advantages of the present solutions is that they can easily and accurately be numerically evaluated very close to the sharp wavefront discontinuity.

In Figs. 3a–3f the exact solution to the pulsed, point source problem

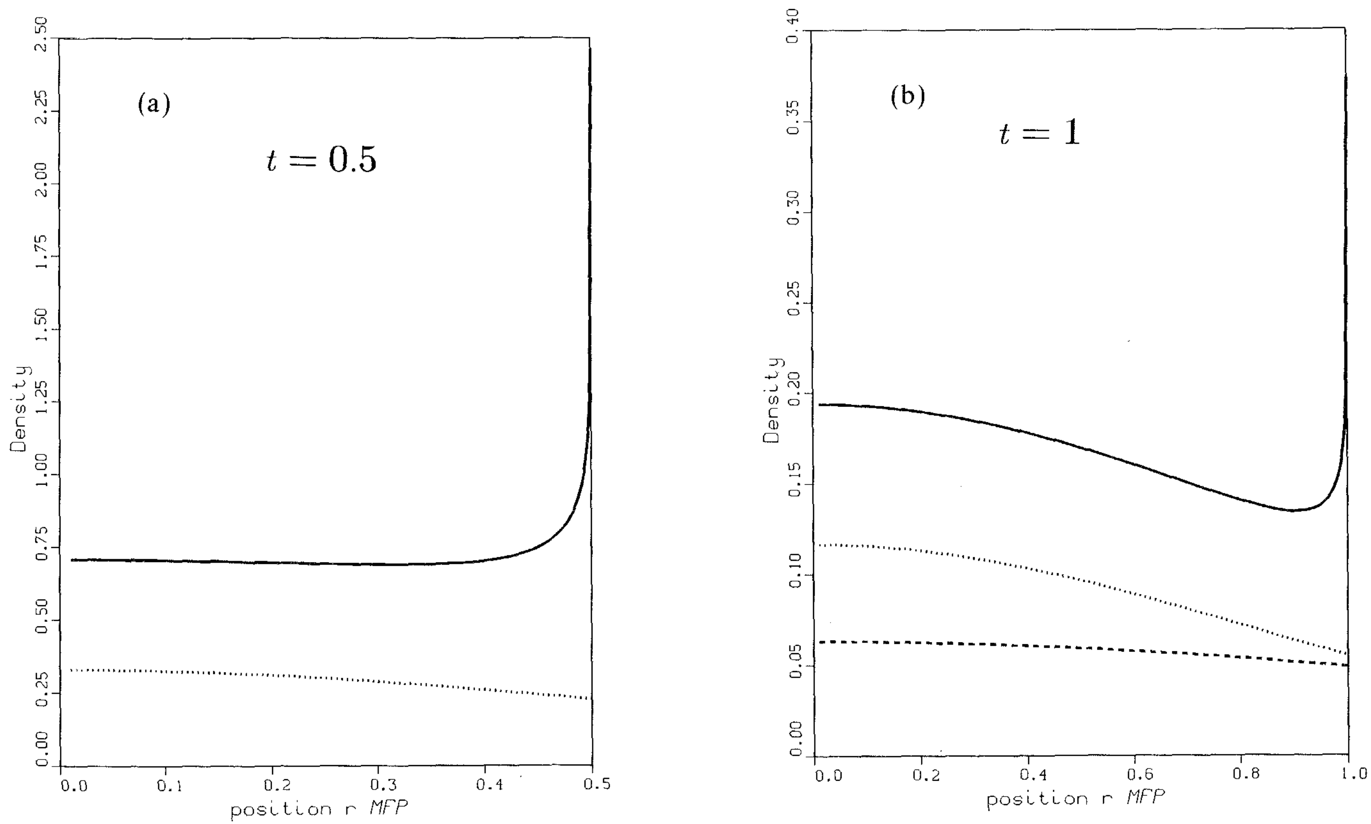

Fig. 3. The exact density profile (solid line) due to a pulsed, point source compared with the hydrodynamic pole solution (dashed line) and the simple diffusion theory result (dotted line) for a sequence of times $t = 0.5$ (a), 1 (b) 2 (c), 3 (d), 5 (e), 18 (f). (Time is measured in MFT, particle velocity $v = 1$, and position r is measured in MFP.)

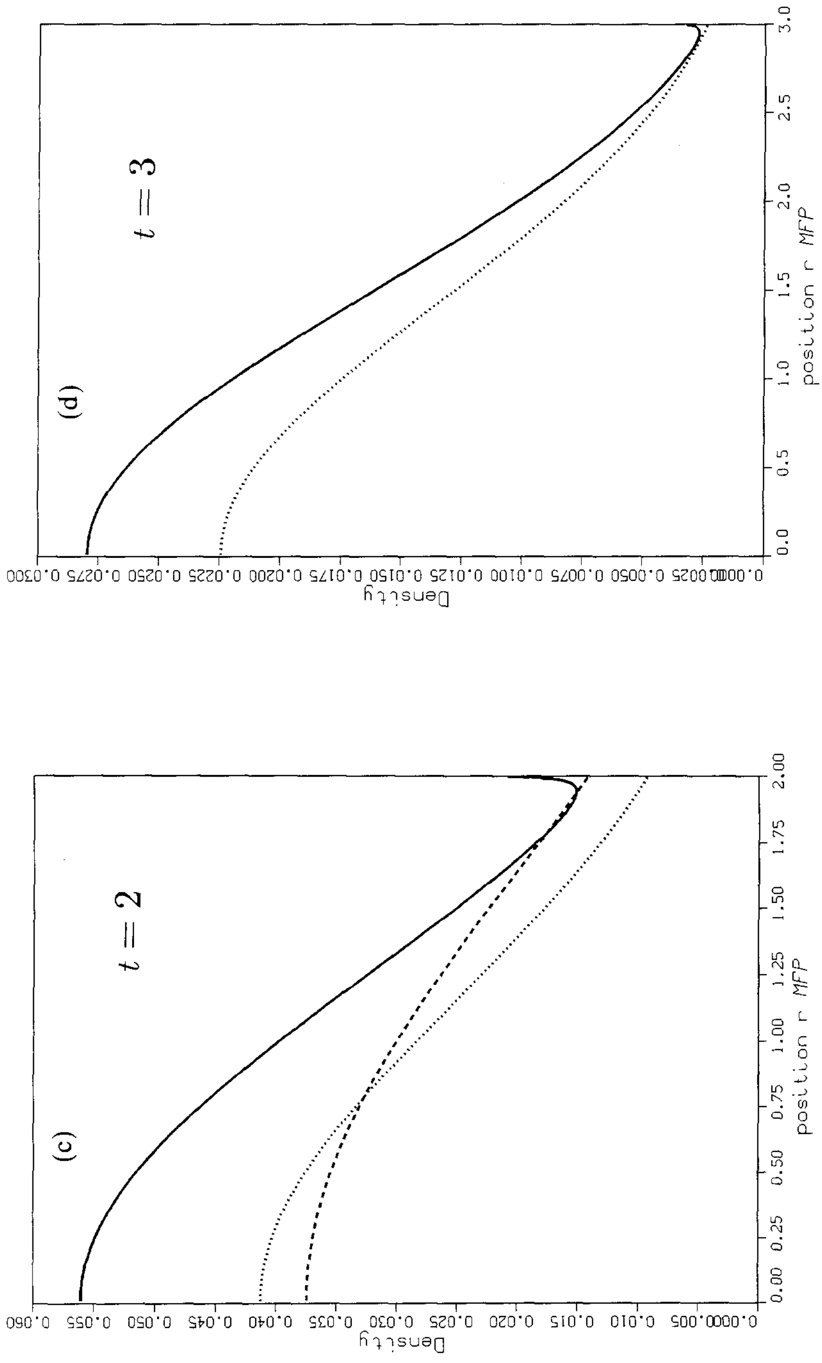

Fig. 3. (*Continued*)

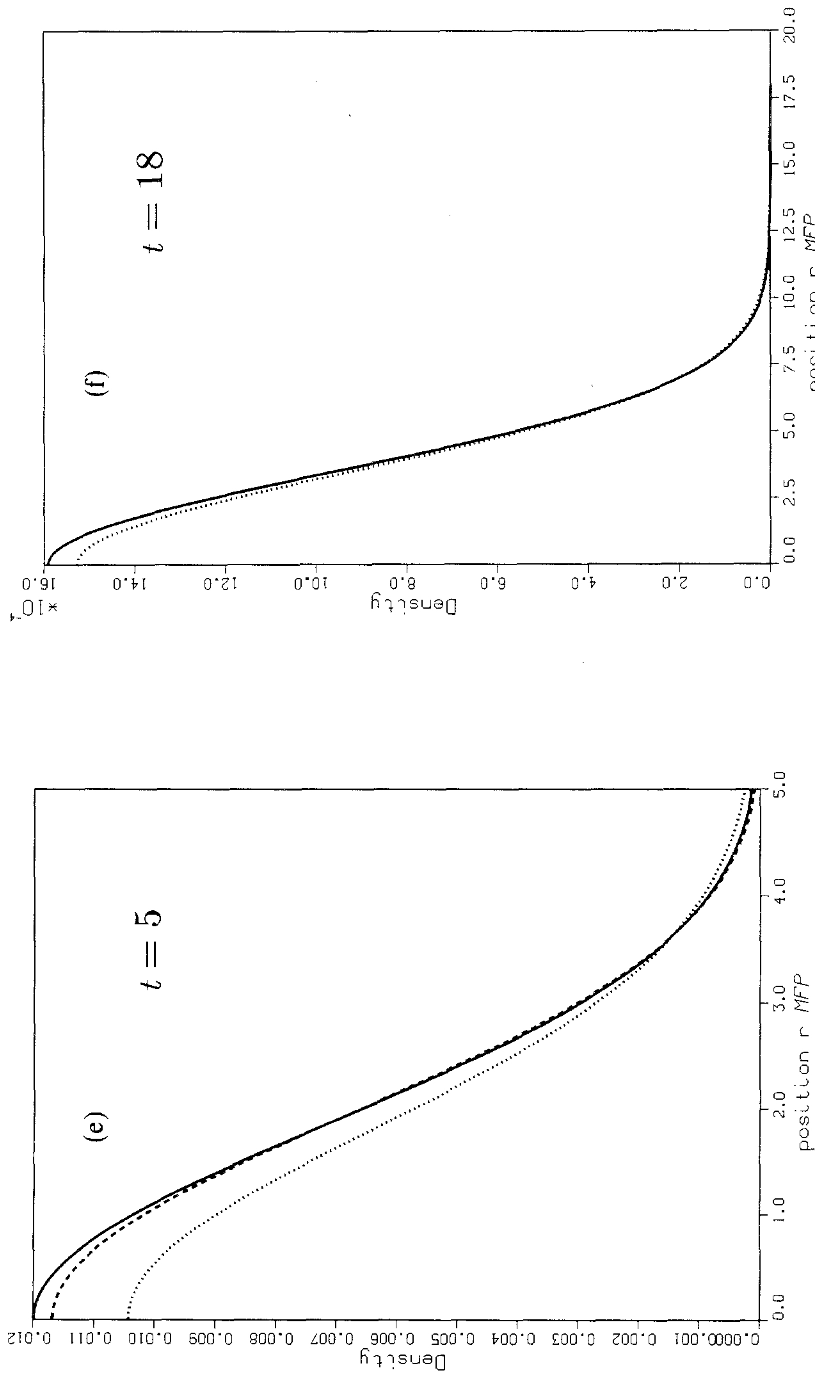

Fig. 3. *(Continued)*

$\rho_{\text{pt}}(r, t)$ is plotted as a function of r (measured in MFPs) for a sequence of times t (measured in MFTs). Also plotted are

$$\rho_{\text{diff}}(r, t) = \frac{2\pi}{(4\pi Dt)^{3/2}} e^{-r^2/4Dt} \tag{50}$$

the diffusion theory (hydrodynamic) result, and in some of the figures the hydrodynamic pole solution $\rho^{\text{pt}}_{\text{pole}}(r, t)$ [Eq. (48)].

The sequence of density profiles vividly depicts how at early times ($t \lesssim \tau$) the transport transients make a large contribution to the density, especially near the sharp, shell-like wavefront moving out "ballistically" as $r \propto t$. The hydrodynamic pole contribution plays a large role even at intermediate times $t \sim \tau$, but, as it simply falls off monotonically with increasing r, it has no interesting structure. As expected, at long times the pole contribution dominates, with the exact result finally relaxing into the simple diffusion result, which expands outward as $\sqrt{t}$, while the shell-like wavefront quickly damps out for times $t \gg \tau$. If there were no collisions, there would only be a delta-function shell propagating out from the origin, so that collisions radically change the nature of the dynamics. Furthermore, a simple RTA to the collision integral [see Eq. (2)] would only give rise to an expanding, exponentially decaying delta-function shell, which clearly cannot account for the approach to hydrodynamics.

From Figs. 3a–3f, we see that at early times the hydrodynamic pole contribution underestimates both the exact result and the simple diffusion result. The simple diffusion solution contains the unphysical result that the density is finite even for $r > t$, in contrast to the exact result, which is discontinuous at $r = t$ and identically zero for $r > t$. Since both the exact and diffusion results conserve the total number of particles, the diffusion result needs those particles in the unphysical region to get the correct total number of particles. We also see that the exact pole contribution eventually crosses over the diffusion result (see Fig. 3c) in its evolution toward the exact result. At very early times, $t \ll \tau$, both the diffusion result and the exact result are flat right up to the wavefront, and the diffusion result can underestimate the exact result by many orders of magnitude (e.g., 3 at $t = 0.01$).

By $t = 5$ (Fig. 3e) the density has more or less relaxed into the simple diffusion form, with the expected result that at late times the region of greatest discrepancy between the exact and diffusion results is near the origin (this is true because the diffusion result is obtained from the exact pole solution in the limit $r \gg l$ and $t \gg \tau$). Near the edge of the wavefront at late times the diffusion result actually appears to slightly *overestimate* the exact result. By $t=18$ (Fig. 3f) the exact result has clearly lost its propagating nature and the width of the expanding disturbance is no longer $\propto t$. The small deviation near the origin gets damped out completely for $t \gg \tau$.

The density for an arbitrary, time- and position-dependent source can be obtained using Eq. (42) [the source must be turned on, $S(r, t)=0$ for $t<0$, and localized, $S(r, t)=0$ for $r>r_{\max}$]. As an illustration, Figs. 4a–4e and 5 show the density due to a distributed, pulsed source, which is just an initial-value problem with initial conditions for the density

$$\phi(r, 0)=\begin{cases} 1 & \text{for} \quad r \leqslant r_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

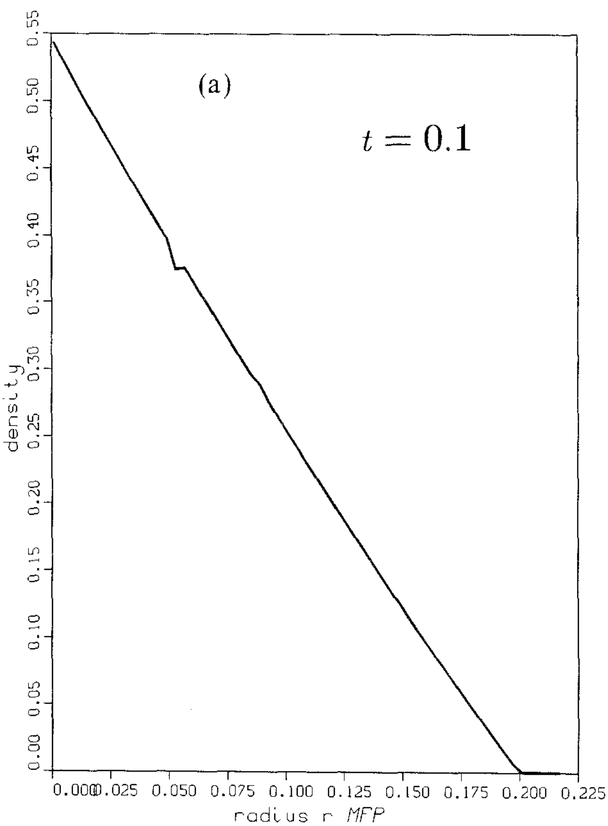

Fig. 4. The exact result for the density profile (solid line) due to an extended (flat), pulsed source of radius $r_{\max}=0.1$ for a sequence of times $t=0.1$ (a), 1 (b) 2 (c), 3 (d), 5 (e). Dotted line in panel (b) is from a numerical computation using the TIMEX transport code (see text). (Time t is measured in MFT, particle velocity $v=1$, and position r is measured in MFP.)

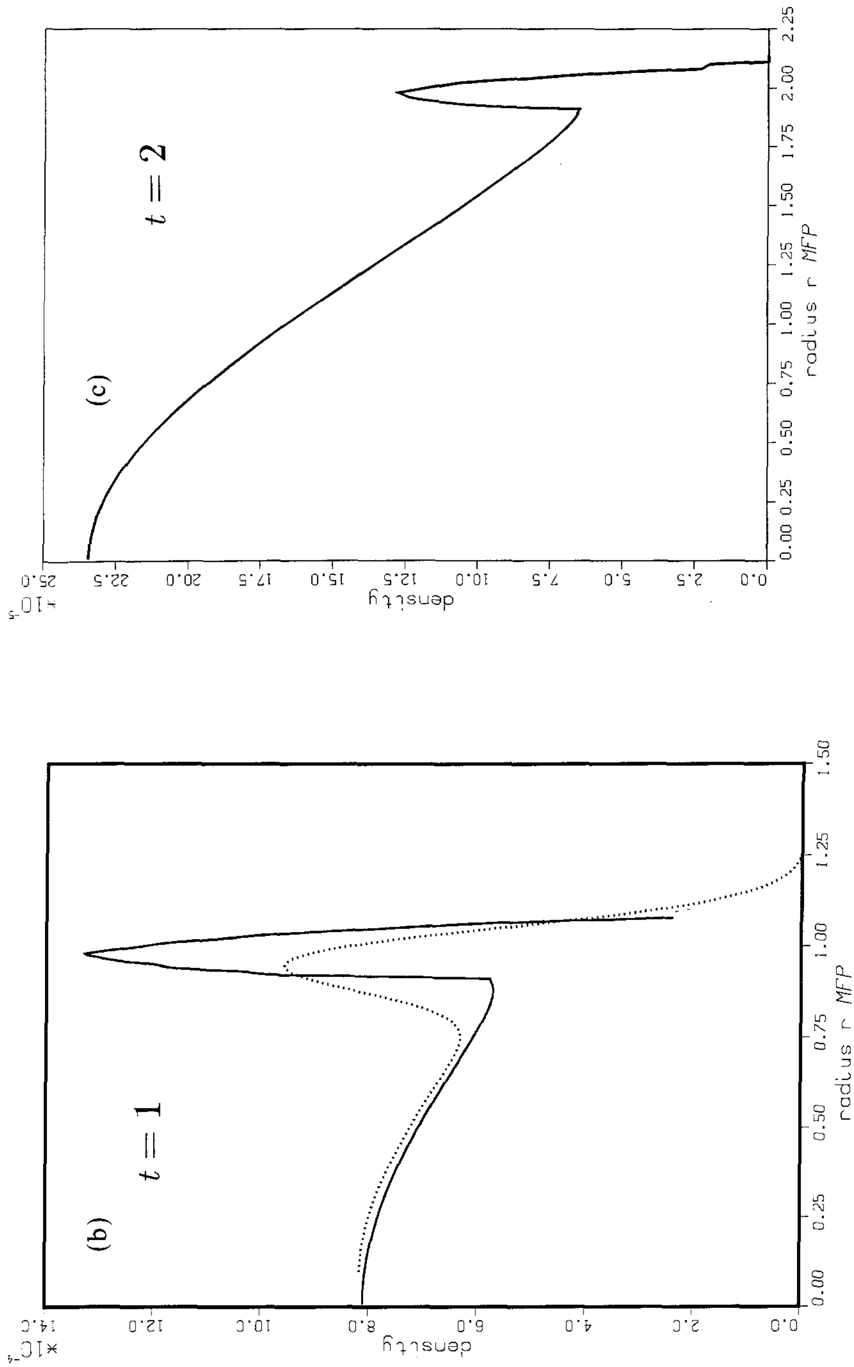

Fig. 4. (*Continued*)

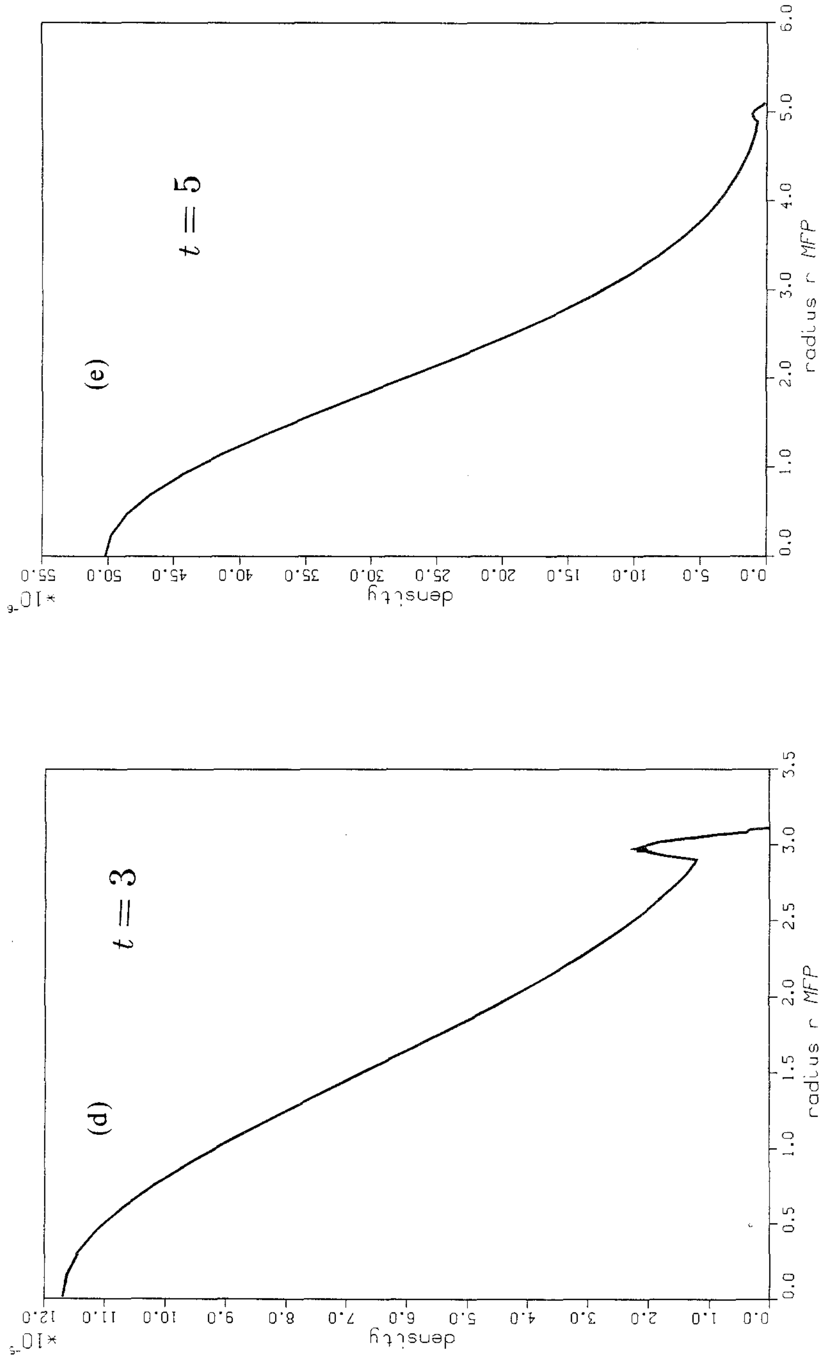

Fig. 4. *(Continued)*

The width of the initial distribution introduces a new length scale r_{max}, and we might expect qualitatively different results depending on whether $r_{max} <$ or > 1 (units in MFP). Intuitively, we might expect that a shell-like wavefront (similar to the one found in the point source problem) can form for roughly $r_{max} < 1$, since then the system is still in the kinetic regime when the wavefront shell is beginning to emerge from the region of the initial disturbance, but not form for (roughly) $r_{max} > 1$, since the system will essentially be in the hydrodynamic regime at the time the wavefront shell ought to be emerging. These ideas are borne out by comparing the sequence of density profiles displayed in Figs. 4a–4e and Fig. 5.

I first discuss the results for $r_{max} = 0.1 \ll 1$. These results are qualitatively the same as in the point source problem, but now the shell-like wavefront propagating out at a speed approximately equal to the speed of the particles is more highly visible. At the earliest time $t = 0.1$ (Fig. 4a) only the slightest hint of the formation of a shell structure can be seen, while at $t = 1$ the shell structure is prominent and well developed. In Fig. 4b, I have also plotted the density found by solving the Boltzmann

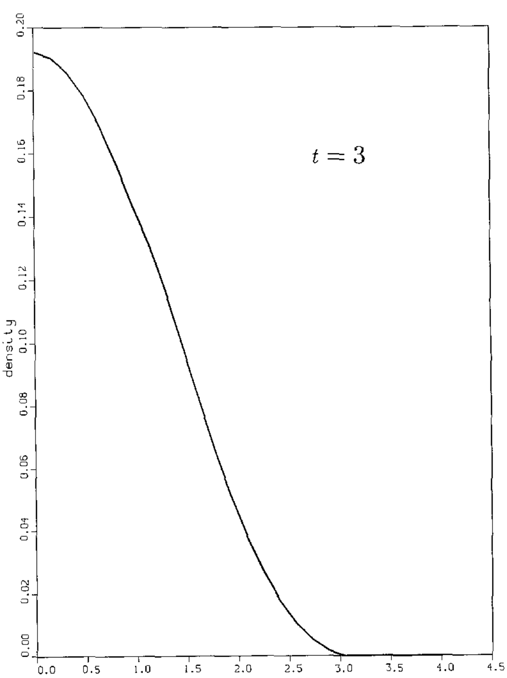

Fig. 5. The exact density profile for an extended, flat pulsed source with $r_{max} = 1$ for time $t = 3$; no shell-like wavefront is visible here, nor at earlier or later times either.

equation numerically using the transport code TIMEX,[23] which employs standard numerical methods to solve the integrodifferential equation. Even for fairly fine space and time meshes (0.01 MFP and 0.005 MFT, respectively) TIMEX has an extremely difficult time following the sharp wavefront, although the code does well in the region behind the shell. As time progresses (see Figs. 4c–4e) the sharp wavefront structure begins to get damped out as collisions depopulate the shell, and for times $t > 5$, the shell has all but disappeared and the system has just about completely entered the hydrodynamic regime.

In Fig. 5, I show the results for $r_{\max} = 1$ and $t = 3$. In line with the earlier discussion, no shell structure appears to form.

4. DISCUSSION OF THE EVOLUTION OF A LOCALIZED DISTURBANCE

The results obtained in the previous section lead to some general conclusions concerning the temporal evolution of a localized disturbance.

One of the important results is that the qualitative character of the density profile depends critically on the ratio $r_{\max}$/MFP, where in general $r_{\max}$ is the characteristic length scale of the initial disturbance. For roughly $r_{\max} <$ MFP a sharp ballistic wavefront shell structure develops, superimposed on top of a smooth background. The structure persists for a few MFTs, but then quickly dies out as the disturbance relaxes into a diffusive form; this shell structure is due to the transport transient parts of the density. Also, at early times the hydrodynamic pole piece is a *worse* approximation to the exact result than the simple diffusion result, while at later times the reverse is true, and all three solutions converge for $t \gg \tau$.

For roughly $r_{\max} >$ MFP no sharp wavefront shell structure develops, and there is only a smooth, monotonically decaying density profile.

The results for the point source problem show explicity how badly both the RTA solution (a delta-function shell for a point source) and the exact solution to the generalized hydrodynamic equation approximate the exact solution for a localized disturbance in the kinetic regime: the hydrodynamic approximations fails completely to account for the sharp ballistic shell that develops in some circumstances, and the RTA partially accounts for the shell, but not the smooth background and the approach to hydrodynamics. The appeal of the present results is that general qualitative ideas concerning the transition to a hydrodynamic description after a severe disequilibration have been backed up by explicit calculations.

5. RESPONSE OF A CHARGED GAS TO A CONSTANT APPLIED ELECTRIC FIELD

An important model problem in charged particle transport concerns the response of a uniform system to an externally applied, constant electric field **E**. This might model carrier transport in a semiconductor, for example. If only elastic scattering processes are considered (say carrier–impurity scattering), then the simple Lorentz gas BE can be used to describe transport phenomena in these systems.

As emphasized by Mattis *et al.*,[24] if there are no inelastic scattering processes, then it is essential to solve the full time-dependent problem, because there is no steady-state solution. Even though there is no steady-state solution, it can be shown[24] that (i) the time-dependent BE implies that Ohm's law [$\mathbf{j} = \sigma_c \mathbf{E}$, where $\mathbf{j} = \int d^3p\, \mathbf{p}\psi(\mathbf{p}, t)$ is the current density] is obeyed after a transient period during which the current is built up and (ii) the Joule heating expression for the energy density $\partial U/\partial t = \mathbf{E} \cdot \mathbf{j}$ holds for all times; in addition, the BE leads to a dc conductivity with the usual Drude form, $\sigma_c = ne^2\tau/m$ (where n is the density of particles), so it is clear that the simple Lorentz gas BE contains many realistic features, making it a worthwhile model to study.

Unfortunately, solving even this simplified model is very difficult, and there are few exact solutions. Ganapol and Boffi[25] showed that with a few restrictions the applied field problem can be mapped onto the spherical geometry localized disturbance problem. Then, using the multiple-collision formalism originally devised for the latter problem, they were able to solve some simplified isotropic scattering, applied field problems. More recently, Mattis *et al.* have solved the applied field problem in the highly anisotropic forward/backscattering limit (particles can only scatter from $\hat{\mathbf{p}} \to \pm\hat{\mathbf{p}}$), which allowed the BE to be reduced to a partial differential equation. The purpose of this section is to show that the exact solutions to the localized disturbance problem derived above can be used to find new exact solutions to the isotropic scattering, applied field problem.

In the presence of a constant, uniform electric field the BE for a spatially uniform, charged Lorentz gas becomes (assuming isotropic scattering)

$$\frac{\partial \psi}{\partial t} + \mathbf{F} \cdot \nabla_{\mathbf{p}} \psi + v\sigma\psi = v\sigma \int \frac{d\hat{\mathbf{p}}'}{4\pi} \psi(p, \hat{\mathbf{p}}', t) + S(p, \hat{\mathbf{p}}, t) \tag{52}$$

where $\mathbf{F} = e\mathbf{E}$, $\mathbf{p} = m\mathbf{v}$ is the momentum, and $p = |\mathbf{p}|$. By going to spherical coordinates in momentum space and choosing the z-axis along **E**, the BE (52) becomes

$$\frac{\partial \psi}{\partial t} + F\mu \frac{\partial \psi}{\partial p} + F \frac{1 - \mu^2}{p} \frac{\partial \psi}{\partial \mu} + v\sigma\psi = \frac{v\sigma}{2} \int_{-1}^{+1} d\mu'\, \psi(p, \mu', t) + S(p, t) \tag{53}$$

where μ is now defined to be the direction cosine of the velocity with the electric field vector (i.e., $p_z = p\mu$). The system is taken to be in global thermal equilibrium for times $t < 0$, and the force is turned on at $t = 0$. A comparison of Eqs. (40) and (53) shows that the applied field problem maps onto the localized disturbance problem provided that $r \leftrightarrow p$, $v \leftrightarrow F$ (in the gradient terms only), and $\lambda \equiv v\sigma(v)$ is independent of v.

In contrast to what happens in the localized disturbance problem, the magnitude of the momentum (p) now enters in a crucial way: the applied force couples states with different energies, but because the system is initially assumed to be uniform, there is no longer any spatial dependence to the problem. In the localized disturbance problem the distribution function spreads out in real space as the particles stream along, whereas in the present problem the distribution function spreads out in momentum space as the constant force accelerates the particles.

Dimensionless time and momentum variables can be defined by $t \to t\lambda$ and $p \to p\lambda/F$, and a modified source term by $S \to S/\lambda$. Then, given the source term, the density of particles with momentum magnitude p (independent of direction), $\phi(p, t) = \int_{-1}^{+1} d\mu'\, \psi(p, \mu', t)$, can be found from the localized disturbance result (42),

$$\phi(p, t) = 4\pi \int dp'\, p'^2 \int dt'\, \rho_{\mathrm{ss}}(p, t; p', t')\, S(p', t')$$

where ρ_{ss} denotes the density due to an isotropic, pulsed shell source in momentum space located at $p = p'$ and triggered at $t = t'$ [i.e., $S(p, t) = (1/4\pi p'^2)\, \delta(p - p')\, \delta(t - t')$], where

$$\rho_{ss}(p, t; p', t') = \frac{1}{4\pi p' p} \left[\rho_{\mathrm{pl}}(|p - p'|, t - t') - \rho_{\mathrm{pl}}(p + p', t - t')\right]$$

with $\rho_{\mathrm{pl}}(p, t)$ the plane symmetry density GF defined in Eqs. (33)–(35).

To get the full distribution function, the BE (52) is rewritten (in the dimensionless time and momentum variables, and with the modified source term) as

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial p_z} + 1\right] \psi(\mathbf{p}, t) = Q(p, t) \tag{54}$$

where $Q(p, t) \equiv \frac{1}{2}\phi(p, t) + S(p, t)$ is a *known* source term (the density is assumed to have been calculated in the manner described above).

If $\mathbf{p} = (\mathbf{p}_\perp, p_z)$, then $Q(p, t) = Q((p_\perp^2 + p_z^2)^{1/2}, t)$ and $\psi = \psi(\mathbf{p}_\perp, p_z, t)$.

To solve Eq. (54), it is convenient to introduce the GF $G(p_z, t; p'_z, t') = G(p_z - p'_z, t - t')$, which satisfies

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial p_z} + 1\right] G(p_z, t; p'_z, t') = \delta(p_z - p'_z)\, \delta(t - t') \tag{55}$$

and has the form

$$G(p_z, t; p'_z, t') = e^{-(t-t')} \delta(p_z - p'_z + t - t')$$

Then the linearity of Eq. (55) implies that the solution to Eq. (54) is

$$\psi(\mathbf{p}_\perp, p_z, t) = \int dt' \int dp'_z\, G(p_z, t; p'_z, t')\, Q((p'^2_\perp + p'^2_z)^{1/2}, t')$$

$$= \int dt'\, e^{-(t-t')} Q(\{p'^2_\perp + [p'_z - (t - t')]^2\}^{1/2}, t')$$

If there were no scattering, then for an initial value problem with the source $S(p, t) = \delta(t)\, S_0(p)$ [i.e., $\psi(\mathbf{p}_\perp, p_z, t = 0) = S_0(p)$], the solution to Eq. (54) reduces to $\psi(\mathbf{p}_\perp, p_z, t) = S_0((p^2_\perp + [p_z - t]^2)^{1/2})$, which makes perfect physical sense, since the number of particles with $\mathbf{p} = (\mathbf{p}_\perp, p_z)$ at time t is given by the number of particles that started out at time $t = 0$ with momentum $p_z - Ft$ [the particles are simply accelerating, so that $p_z(t) = p_z(0) + Ft$].

If the system is in thermal equilibrium for time $t < 0$ with the Maxwell–Boltzmann distribution $\psi_0(p) = C \exp(-p^2/2mk_B T)$, then the solution for times $t > 0$ after the force is turned on is

$$\phi(p, t) = 4\pi C \int dp'\, p'^2 \rho_{ss}(p, t; p', 0) \exp(-p'^2/2mk_B T) \tag{56}$$

and

$$\psi(\mathbf{p}_\perp, p_z, t) = \int_0^t dt'\, \{\exp[-(t - t')]\, \phi(\{p'^2_\perp + [p'_z - (t - t')]^2\}^{1/2}, t')\}$$
$$+ C \exp(-t) \exp(-p^2_\perp/2mk_B T) \exp[-(p_z - t)^2/2mk_B T] \tag{57}$$

The density can be obtained numerically, but I shall not present any detailed numerical results here. The qualitative nature of the solution, at least at long times $t \gg \tau$, is just that obtained in ref. 24: if one plots the distribution function $\psi(\mathbf{p}_\perp, p_z, t)$ for fixed $\mathbf{p}_\perp$ as a function of p_z, then at

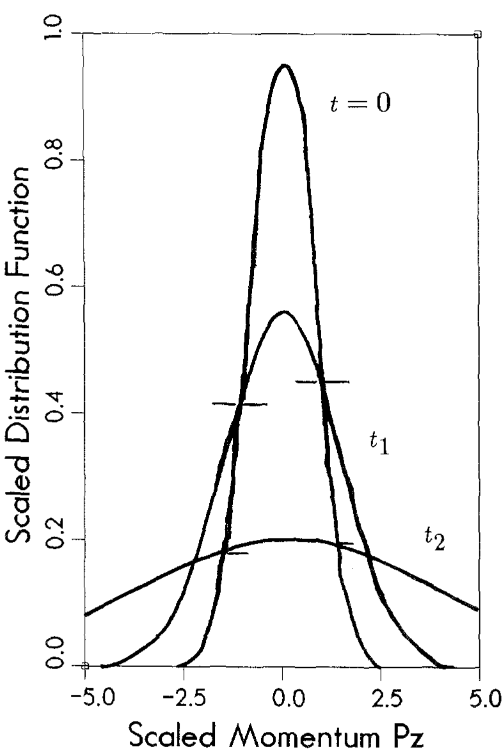

Fig. 6. Qualitative form for the distribution function $\psi(\mathbf{p}_\perp, p_z, t)$ as a function of p_z for fixed $\mathbf{p}_\perp$ for times $0 < t_1 < t_2$ long after the transport transients have died out, $t_1 \gg \tau$.

times long after the transport transients have died out one expects that an initial Gaussian Maxwell–Boltzmann distribution gets squashed and spreads out as a function of p_z with a slight asymmetry developing (which gives rise to a finite current) as the gas heats up (see Fig. 6). A problem similar to the one studied in this section has been discussed in great detail in ref. 24, and since the present results appear to be qualitatively the same as the ones obtained there, I refer the reader to this reference for a more complete discussion of the ramifications of the exact solutions to the BE for a charged gas.

6. SUMMARY AND DISCUSSION

I have presented some exact solutions to two important classes of problems in nonequilibrium statistical mechanics. My aim was to use the relatively simple Lorentz gas model as a testing ground for studying the relationship between kinetic theory and hydrodynamics in a setting where exact results could be obtained. I observed that the exact solutions are relatively complicated even for the simple Lorentz gas model, with transport transient propagating modes appearing as well as a generalized diffusive, or hydrodynamic, part that reduces to the well-known simple diffusion theory result at long times and long wavelengths. Using the exact solutions, I have generated a sequence of density profiles that follow a pulsed, localized disturbance from an early-time kinetic regime where the propagating parts of the solution are the most important through a crossover regime $t \sim \tau$ to a macroscopic hydrodynamic regime, where the simple diffusion equation gives an accurate description of the evolution of the nonequilibrium density. Furthermore, I have shown how simplified approximations such as hydrodynamic treatments and RTAs to the collision integral fail to account for important features of particle transport. The exact results for the Lorentz gas problems were then used to gain some insight into the general problem of the decay of a localized disturbance in a system initially in global equilibrium and the subsequent approach to hydrodynamics. In particular, I have shown graphically how nonhydrodynamic shell-like structures can form under certain initial conditions; I have also shown how long and to what extent they can survive before they are more or less completely damped out by collisions. I have also obtained some exact results for the nonequilibrium particle distribution induced by an applied electric field turned on at $t = 0$.

The relative paucity of exact solutions to even simplified model Boltzmann transport equations makes any new exact solutions a valuable contribution to the treasure chest of exactly soluble problems. This is especially true for the types of difficult *time- and space*-dependent problems treated here (i.e., the approach to hydrodynamics), for which, to my knowledge, there are no other known exact results (cf. ref. 19a). In the field of transport theory there is an abundance of research on the formal mathematical properties of kinetic equations on the one hand, and a body of semiheuristic, often inadequately justified, work on the transport properties of real, complicated physical systems on the other, and I believe the exact results presented here can be useful in attempts to bridge this gap (cf. ref. 18). I hope that the exact results discussed here have helped the reader to gain some mathematical and physical insight into the general problem of how a macrodescription of nonequilibrium states of matter evolves from a micro one.

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